Computing Dimensionally Parametrized Determinant Formulas

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Abstract: We are interested in dimensionally parametrized determinant formulas for specially structured matrices. Applications of this question occur for example in the study of arbitrary dimensional geometric predicates. We will investigate determinant formulas for a number of important matrix classes and discuss the implementation of Maple packages that automatically derive the determinant formula for specified matrices of these classes.

Introduction

Determinants have a long history in mathematics and arise in numerous applications. Consequently, they have been researched extensively which has yielded efficient algorithms for determinant computation. Here we are not interested in the *value* of a determinant of fixed integer order but rather in the determinant *formula* of a specially structured matrix of symbolic dimension n. It is assumed that a certain simple structure of a matrix yields a corresponding special structure of its determinant formula. This problem has been investigated very early; Muir's "Treatise on the Theory of Determinants"[4] contains a large number of early papers related to this subject motivated by applications in algebra and analysis or simply by the interesting structure of the matrix coefficients. Nowadays, many of these results tend to be forgotten or are buried in induction exercises in linear algebra books. Applications of dimensionally parametrized determinant formulas occur for example in the study of arbitrary dimensional geometric predicates in determinant form: If we want to prove a general statement for a special configuration then we need the determinant formula of the predicate.

In the following sections we will investigate determinant formulas for three important matrix classes, the Frameforms, the Alternants and the Continuants. Moreover, we will discuss the implementation of Maple packages that allow a specification of matrices of these classes and automatically derive its determinant formula.

Frameforms

We will first examine a matrix class where only the bordering rows and columns as well as the main diagonal may contain nonzero entries. Matrices of this class will be called *frameforms*.

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Motivation

Geometric predicates such as the sidedness test – do $d+1$ points of $\mathbb{R}^d$ lie on a common hyperplane ? – or the in-sphere test – do $d+2$ points of $\mathbb{R}^d$ lie on a common sphere ? – may be written in determinant form [2].

Consider the following example in [2]: We have a configuration of $d+2$ points in $\mathbb{R}^d$, two distinct points $\overline{s} = (s, \ldots, s)$ and $\overline{t} = (t, \ldots, t)$ from the main diagonal and one point $\overline{t_i} = t_i \cdot e_i$ from each axis.

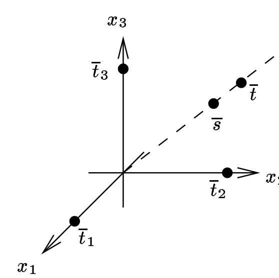

Figure 1: Configuration in $\mathbb{R}^3$

This point configuration results in the following in-sphere determinant:

$$S = \begin{vmatrix} 1 & t & t & \cdots & t & dt^2 \\ 1 & t_1 & 0 & \cdots & 0 & t_1^2 \\ 1 & 0 & t_2 & \ddots & \vdots & t_2^2 \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 1 & 0 & \cdots & 0 & t_d & t_d^2 \\ 1 & s & s & \cdots & s & ds^2 \end{vmatrix}_{d+2} .$$

We are looking for an easy dimensionally parametrized determinant formula such that we can show that the determinant does not vanish for given ranges of the entries which would establish that the $d+2$ chosen points are not cospherical. The determinant S is in frameform and we will show in the sequel how to establish its determinant formula.

Determinant Formulas for Frameform Matrices

We will show that it is possible to derive the most general form of frameform matrices from simpler forms.

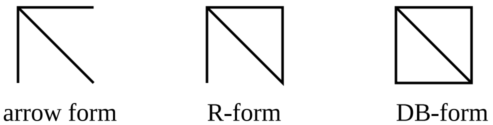

Figure 2: Nonzero shapes of different frameforms

We will proceede as follows: First we will give a dimensionally parametrized determinant formula for arrow forms which will allow us a straightforward generalization to R-forms. The most general form, the DB-form, will be obtained by a combination of R-forms.

Arrow forms

The only nonzero elements of an arrow form determinant are located in the first row, the first column or the main diagonal. It will be denoted as $\text{ARROW}_n(r_1, c_1, d)$:

$$\begin{vmatrix} d(1) & r_1(2) & r_1(3) & \cdots & r_1(n) \\ c_1(2) & d(2) & 0 & \cdots & 0 \\ c_1(3) & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & d(n-1) & 0 \\ c_1(n) & 0 & \cdots & 0 & d(n) \end{vmatrix}_n .$$

Expansion of the first row and clever restructuring establishes the following determinant formula:

$$\text{ARROW}_n(r_1, c_1, d) = \prod_{l=1}^{n} d(l) - \sum_{l=2}^{n} r_1(l) c_1(l) \prod_{\substack{k=2 \\ k \neq l}}^{n} d(k).$$

R-forms

In comparison to arrow forms, the last column may also contain nonzero elements in R-forms.

$$\begin{vmatrix} c_1(1) & r_1(2) & \cdots & \cdots & r_1(n-1) & c_n(1) \\ c_1(2) & d(2) & 0 & \cdots & 0 & c_n(2) \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ c_1(n-1) & 0 & \cdots & 0 & d(n-1) & c_n(n-1) \\ c_1(n) & 0 & \cdots & 0 & 0 & c_n(n) \end{vmatrix}$$

An R-form will be denoted as $\text{RFORM}_n(r_1, c_1, c_n, d)$. Expanding the last row it may be seen that an R-form determinant formula can be obtained from two arrow forms of lower order.

$$\begin{aligned} \text{RFORM}_n(r_1, c_1, c_n, d) = &\; c_n(n)\text{ARROW}_{n-1}(r_1, c_1, d) \\ - &\; c_1(n)\text{ARROW}_{n-1}(\widetilde{r}_1, \widetilde{c}_n, \widetilde{d}) \end{aligned}$$

where $\widetilde{r}_1, \widetilde{c}_n, \widetilde{d}$ are new generating functions obtained by swapping column 1 and $n-1$ in the corresponding minor R_{n1} to get arrow form.

DB-forms

Now we turn to the most general case of frameforms, i.e. allowing nonzero elements in all bordering rows and columns and the main diagonal. A general DB-form will be denoted as $\text{DBFORM}_n(r_1, r_n, c_1, c_n)$:

$$\begin{vmatrix} c_1(1) & r_1(2) & \cdots & \cdots & r_1(n-1) & c_n(1) \\ c_1(2) & d(2) & 0 & \cdots & 0 & c_n(2) \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ c_1(n-1) & 0 & \cdots & 0 & d(n-1) & c_n(n-1) \\ c_1(n) & r_n(2) & \cdots & \cdots & r_n(n-1) & c_n(n) \end{vmatrix}$$

Expanding the last row of such a general DB-determinant, we see after some restructuring that it is possible to express its formula as a combination of R-forms.

$$\begin{aligned} \text{DBFORM}_n(r_1, r_n, c_1, c_n) &= \text{RFORM}_n(r_1, c_1, c_2, d) \\ &- \sum_{l=2}^{n-1} r_n(l)\text{RFORM}_{n-1}(\widehat{r}_1, \widehat{c}_1, \widehat{c}_n, \widehat{d}) \end{aligned}$$

where $\widehat{r}_1, \widehat{c}_1, \widehat{c}_n, \widehat{d}$ are new generating functions obtained by swapping row l down to the bottom in the corresponding minor DB_{nl} to get R-form.

Generalizations

It remains to note that it is easy to transform similar shapes like arrows pointing to the bottom right corner into the discussed standard shapes.

So far we also required that the nonzero elements should reside in bordering rows and columns. However, it is straightforward to show that we may drop this assumption since it is possible to obtain this bordering form via pairwise swappings. Refer to [5] for details.

A Maple package for frameform determinants

We have implemented a Maple package that enables the user to specify a general frameform determinant and automatically computes the corresponding determinant formula using the preceding results. The package works as follows: First, the specification is parsed and tested for correctness, then a transformation into standard form is performed and finally the corresponding formula is applied after trying out simplifications. Features are options that enable the display of the specified dimensionally parametrized determinant (using "o"s as dots for illustration) and that check the computed formula via substitution of integer orders and comparison with the normally computed determinant. Details can be obtained from [5] or the online help pages.

Specification

The specification of a frameform determinant is designed to be a list of row, column or diagonal specifications:

`specDet=[ specL_1, ... , specL_k ]`

In this list we have lists of tuples of the form

`specL=[ type[pos], fctn_specL ]`

specifying a certain determinant chunk. Here `type` may be `row`, `col`, `diag` and `pos` can be between `1` and `n` for `row` and `col` or `0` for `diag`.

`fctn_specL` is the piecewise specification of the elements of the current matrix chunk and has the form

`[[intv_1, fctn_1(i)],...,[intv_k,fctn_k(i)]]`.
A restriction is that piecewise specifications are only possible in the ranges `[1..p1]` and `[n-p2..n]`. If no piecewise specification is needed then `fctn_specL` may be simply a function `f` possibly dependent on `i`.

Other shortcuts like omitting double specification of overlapping corner elements are also possible (refer to the online help pages or [5] for details).

Examples

```
> with(FRAMEFORMS):

> Arrow(n,[ [row[1],a], [col[1],[2..n,b]],
>    [diag,[2..n,1]] ],print,check);
```

Matrix :

$$\begin{bmatrix} a & a & a & o & o & o & a \\ b & 1 & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 1 & 0 & 0 & 0 & 0 \\ o & 0 & 0 & o & 0 & 0 & 0 \\ o & 0 & 0 & 0 & o & 0 & 0 \\ o & 0 & 0 & 0 & 0 & o & 0 \\ b & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Determinant :

$$n >= 2$$

$$a - b\,a\,n + b\,a$$

```
> DBform(d+2,[ [col[1],1], [col[d+2],
>    [[1,d*t^2],[2..d+1,t[i-1]^2],[d+2,d*s^2]]],
>    [diag,[2..d+1,t[i-1]]], [row[1],[2..d+1,t]],
>    [row[d+2],[2..d+1,s]] ],print,check);
```

Matrix :

$$\begin{bmatrix} 1 & t & t & o & o & o & t & d\,t^2 \\ 1 & t_1 & 0 & 0 & 0 & 0 & 0 & {t_1}^2 \\ 1 & 0 & t_2 & 0 & 0 & 0 & 0 & {t_2}^2 \\ o & 0 & 0 & o & 0 & 0 & 0 & o \\ o & 0 & 0 & 0 & o & 0 & 0 & o \\ o & 0 & 0 & 0 & 0 & o & 0 & o \\ 1 & 0 & 0 & 0 & 0 & 0 & t_d & {t_d}^2 \\ 1 & s & s & o & o & o & s & d\,s^2 \end{bmatrix}$$

Determinant :

$$formula\ valid\ for\ ,\ d >= 2$$

$$(1 - \frac{t}{s})\,((\prod_{l\_=1}^{d} t_{l\_})\,d\,s^2 - s\,(\prod_{l\_=1}^{d} t_{l\_})\,(\sum_{l\_=1}^{d} t_{l\_}))$$
$$- (d\,t^2 - d\,s\,t)\left((\prod_{l\_=1}^{d} t_{l\_}) - s\,(\prod_{l\_=1}^{d} t_{l\_})\sum_{l\_=1}^{d} \frac{1}{t_{l\_}}\right)$$

The preceding formula which is the result of our insphere example looks rather nasty. If we take a closer look at it, we see that it may be cleaned up a little bit: Factoring out the nonzero term $(\prod_{l=1}^{d} t_l)/(t-s)$ we get the much nicer formula

$$\sum_{l=1}^{d} t_l \; - dt + dst \sum_{l=1}^{d} \frac{1}{t_l}.$$

Since Maple's `simplify` command does not always produce results which are simplified in our sense, it is suggested that one simplifies the resulting formulas by hand.

Alternants

Let us turn to another important determinant class, the *alternants*. An alternant of order n is a determinant where the entries of the first row are generated by functions $f_1, \dots, f_n$ in one variable x_1, the entries of the second row by the same functions in another variable x_2 and so on. We assume that the column generating functions are multivariate polynomials over a ring.

$$\begin{vmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_n(x_1) \\ f_1(x_2) & f_2(x_2) & \cdots & f_n(x_2) \\ \vdots & \vdots & & \vdots \\ f_1(x_n) & f_2(x_n) & \cdots & f_n(x_n) \end{vmatrix}.$$

The most well known type of an alternant is the Vandermonde determinant, generated by the functions x_i^{j-1} for $i, j = 1, \dots, n$.

$$V = \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix}.$$

A generally known fact is that the formula of the Vandermonde determinant is

$$V = \prod_{1 \le i < j \le n} (x_j - x_i)$$

which is the *difference product* of the variables.

Indeed it is straightforward to show that this difference product appears as a factor of every alternant and that its cofactor is a symmetric function in the variables. How can we compute this cofactor ?

We will present a theorem of [4] that determines the cofactor as a combination of *elementary symmetric functions*. The elementary symmetric function σ_r is the sum of all monomials that are products of r distinct variables:

$$\sigma_r = \sum_{1 \le i_1 < \cdots < i_r \le n} x_{i_1} \cdots x_{i_r}$$

In particular $\sigma_0 = 1$, $\sigma_1 = x_1 + \cdots + x_n$ and $\sigma_n = x_1 x_2 \cdots x_n$. The elementary symmetric functions form a basis of the symmetric polynomials [1].

Theorem

Let $f_j(x_i) = a_{0j} + a_{1j}x_i + a_{2j}x_i^2 + \cdots + a_{rj}x_i^r$ be the column generating functions with $r \ge n-1$ and let $S_k = (-1)^k \sigma_k$.

The cofactor of the difference product of the generated alternant of order n is

$$\begin{vmatrix} a_{01} & a_{11} & \cdots & a_{n1} & a_{n+1,1} & \cdots & a_{r1} \\ a_{02} & a_{12} & \cdots & a_{n2} & a_{n+1,2} & \cdots & a_{r2} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{n0} & a_{n1} & \cdots & a_{nn} & a_{n+1,n} & \cdots & a_{rn} \\ S_n & S_{n-1} & \cdots & S_0 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & 0 \\ 0 & \cdots & 0 & S_n & S_{n-1} & \cdots & S_0 \end{vmatrix}_{r+1}.$$

Proof. See [4] or [5].

At first it seems that we didn't gain anything since we only traded an order n determinant for an order $r+1$ determinant involving coefficients and elementary symmetric functions. However, if we assume $r = n + d$ with $d \in \mathbb{N}$ and only consider monomials as column generating functions, it becomes obvious that only one entry in each of the first n rows is nonzero. This allows easy expansion of the first n rows yielding a minor of order $d+1$ involving elementary symmetric functions. This minor is of integer order and can be computed by standard minor expansion.

Example

Consider the following alternant:

$$A = \begin{vmatrix} 1 & x_1^2 & x_1^3 & \cdots & x_1^n \\ 1 & x_2^2 & x_2^3 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{n-1}^2 & x_{n-1}^3 & \cdots & x_{n-1}^n \\ 1 & x_n^2 & x_n^3 & \cdots & x_n^n \end{vmatrix}_n.$$

The theorem gives us a cofactor determinant

$$\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ S_n & S_{n-1} & \cdots & S_1 & S_0 \end{vmatrix} = (-1)^{n-1} S_{n-1} = \sigma_{n-1}$$

and hence the determinant formula

$$A = \sigma_{n-1} \prod_{1 \le i < j \le n} (x_j - x_i).$$

Using the multilinearity of the determinant we can also handle the case of polynomial generating functions provided that there are only a fixed integer number of them. [5] also discusses some special cases where simple polynomial functions are allowed.

Elementary symmetric polynomials are not the only basis for symmetric polynomials, see [4] and [5] for a discussion of Jacobi's cofactor representation by complete symmetric functions for monomial alternants.

A Maple package for Alternants

We have implemented the discussed approach such that it is possible to obtain the dimensionally parametrized formula of specified alternants meeting the restrictions above. After parsing the specification, the determinant is broken into a combination of monomial alternants whose formulas are determined by computing the cofactor determinant. Since only a fixed integer number of columns may be piecewisely defined, we may simulate the minor expansion of the dimensionally parametrized cofactor determinant. Display and checking facilities are provided as in the frameforms package.

Specification

A piecewise specification is possible according to the frameforms package. Features are discussed in [5] and the online help pages of the package.

Examples

```
> with(ALTERNANT):
```

```
>  Alternant(n,x,[ [1..n-1,x[i]^(j-1)],
>     [n..n,2*x[i]^j-x[i]^(j+1)] ],esf,print);
```

Matrix :

$$\begin{bmatrix} 1 & x_1 & o & o & {x_1}^{(n-2)} & 2\,{x_1}^n - {x_1}^{(n+1)} \\ 1 & x_2 & o & o & {x_2}^{(n-2)} & 2\,{x_2}^n - {x_2}^{(n+1)} \\ 1 & x_3 & o & o & {x_3}^{(n-2)} & 2\,{x_3}^n - {x_3}^{(n+1)} \\ o & o & o & o & o & o \\ o & o & o & o & o & o \\ 1 & x_{n-1} & o & o & {x_{n-1}}^{(n-2)} & 2\,{x_{n-1}}^n - {x_{n-1}}^{(n+1)} \\ 1 & x_n & o & o & {x_n}^{(n-2)} & 2\,{x_n}^n - {x_n}^{(n+1)} \end{bmatrix}$$

Determinant :

formula valid for $, 1 < n$

$$(-\mathrm{S}(1, n, x)^2 + \mathrm{S}(0, n, x)\,\mathrm{S}(2, n, x) - 2\,\mathrm{S}(1, n, x))\; DP(n, x, x_i)$$

Here, $S(k, n, x)$ denotes $(-1)^k \sigma_k(x_1, \ldots, x_n)$ and $DP(n, x, x_i)$ the difference product $\prod_{1 \le i < j \le n} (x_j - x_i)$.

Continuants and Hessenberg Determinants

In this section we will discuss determinant formulas of tridiagonal matrices and Hessenberg matrices. We will show a straightforward recurrence formula for tridiagonal determinants, discuss simplifications and try to extend the results on Hessenberg determinants.

Continuants

The determinant of a tridiagonal matrix is called a *continuant*. Continuants are intimately connected to continued fractions, from which they get their name (see [3]). We will denote the determinant

$$\begin{vmatrix} d_0(1) & d_1(1) & 0 & \cdots & 0 \\ d_{-1}(1) & d_0(2) & d_1(2) & \ddots & \vdots \\ 0 & d_{-1}(2) & d_0(3) & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & d_1(n-1) \\ 0 & \cdots & 0 & d_{-1}(n-1) & d_0(n) \end{vmatrix}$$

with main diagonal generating function d_0 and side diagonal generating functions d_{-1} and d_1 by $C_n(d_{-1}, d_0, d_1)$.

Expanding the last column we obtain a recurrence formula for a general continuant:

$$\begin{aligned} C_n(d_{-1}, d_0, d_1) &= d_0(n) C_{n-1}(d_{-1}, d_0, d_1) \\ &\quad - d_{-1}(n-1) d_1(n-1) C_{n-2}(d_{-1}, d_0, d_1) \end{aligned}$$

with base cases $C_1(d_{-1}, d_0, d_1) = d_0(1)$ and $C_0() = 1$.

Let us take a look at the terms of a general continuant. One term is obviously $d_0(1)d_0(2)\cdots d_0(n)$. The other terms can be obtained from it by replacing any pair $d_0(r)d_0(r+1)$ by $-d_{-1}(r)d_1(r)$ for $1 \le r \le n-1$. This follows since we need one row or column exchange to get $d_{-1}(r)$ and $d_1(r)$ in the position of $d_0(r)$ and $d_0(r+1)$. Iteration of this process eventually derives all continuant terms.

This allows the conclusions that

$$C_n(d_{-1}, d_0, d_1) = C_n(1, d_0, d_{-1}d_1) \qquad (1)$$

and if the main diagonal is 0 :

$$C_n(d_{-1}, 0, d_1) = \begin{cases} (-1)^{n/2} \prod_{i=1}^{n/2} d_{-1}(2i-1)d_1(2i-1) & n \text{ even}, \\ 0 & n \text{ odd}. \end{cases}$$

If the diagonal generating functions of the continuant are constant, say $d_0 = a$, $d_1 = b$ and $d_{-1} = c$, we obtain the general formula

$$C_n(c, a, b) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} a^{n-2k} b^k c^k.$$

We will close the section with a special form of continuants: Consider a continuant where each main diagonal element, except the first and the last one, is the sum of the side diagonal elements of the same row:

$$\begin{vmatrix} a_1+b_1 & b_1 & 0 & \cdots & 0 \\ a_2 & a_2+b_2 & b_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & a_{n-1} & a_{n-1}+b_{n-1} & b_{n-1} \\ 0 & \cdots & 0 & a_n & a_n+b_n \end{vmatrix}$$

Induction establishes the following determinant formula

$$C = \sum_{i=1}^{n} a_1 a_2 \cdots a_{i-1} b_{i+1} b_{i+2} \cdots b_n. \qquad (2)$$

Example

Consider the continuant $C_n(x, 1+x^2, x)$:

$$\begin{vmatrix} 1+x^2 & x & 0 & \cdots & 0 \\ x & 1+x^2 & x & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & x & 1+x^2 & x \\ 0 & \cdots & 0 & x & 1+x^2 \end{vmatrix}$$

Using (1) we get $C_n(x, 1+x^2, x) = C_n(1, 1+x^2, x^2)$, hence we may apply (2) and get the formula

$$C_n(x, 1+x^2, x) = 1 + x^2 + x^4 + \cdots + x^{2n}.$$

Hessenberg Determinants

We will define a general Hessenberg determinant with diagonal generating functions $d_{-1}, d_0, d_1, \ldots, d_{n-1}$

$$\begin{vmatrix} d_0(1) & d_1(1) & \ddots & d_{n-2}(1) & d_{n-1}(1) \\ d_{-1}(1) & d_0(2) & d_1(2) & \ddots & d_{n-2}(2) \\ 0 & d_{-1}(1) & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & d_1(n-1) \\ 0 & \cdots & 0 & d_{-1}(n-1) & d_0(n) \end{vmatrix}$$

by $H_n(d_{-1}, d_0, d_1, \ldots, d_{n-1})$ or briefly $H(n)$.

Again it is possible to derive a recurrence formula for the determinant (see [5] for details) and we obtain

$$H(n) = \sum_{i=0}^{n-1} (-1)^i d_i(n-i) H(n-i-1) \prod_{j=1}^{i} d_{-1}(n-j)$$

with base cases $H(1) = d_0(1)$, $H(0) = 1$ and $H(-s) = 0$ for $s > 0$.

In general this recurrence formula allows no simple closed form but let us take a look at a few special cases: For the sake of simplicity, we assume the diagonal generating functions to be constants.

Consider Hessenberg determinants of the form $H(d_{-1}, d_0, 0, \ldots, 0, d_{n-c}, \ldots, d_{n-1})$ with $c > 0$.
We obtain the formula

$$H(n) = d_0^n + \sum_{k=1}^{c} k(-1)^{n-k} d_0^{k-1} d_{-1}^{n-k} d_{n-k}.$$

See the next subsection for an example.

Above, we discussed the effect of a zero main diagonal on the continuant. We will try to derive a similar result for Hessenberg determinants of the form

$$H_n(d_{-1}, 0, \ldots, 0, d_p, 0, \ldots, 0, d_{n-c}, \ldots, d_{n-1})$$

for $c, p \in \mathbb{N}$.

First we assume $c = 0$ and hence examine the form

$$H_n(d_{-1}, 0, \ldots, 0, d_p, 0, \ldots, 0)$$

We observe that the determinant has to vanish if $n \neq (p+1) \cdot s$ for some multiple s. Otherwise, we can simply expand $H(n) = (-1)^p d_p \prod_{l=1}^{p} d_{-1} H(n-p-1)$ and get

$$H(n) = (-1)^{sp} \prod_{l=0}^{s-1} d_p \prod_{k=1}^{p} d_{-1}.$$

The case $c > 0$ is more tricky, however. It may be observed that $H(0) = 1$, $H(1) = \cdots = H(p) = 0$ and $H(p+1) = d_p \prod_{k=1}^{p} d_{-1}$. Examining the diagonal generating functions $d_{n-c}, \ldots, d_{n-1}$ of the upper right corner of the determinant, we notice that only $d_{n-1}, d_{n-1-(p+1)}, \ldots, d_{n-1-a(p+1)}$ with $a = \left\lfloor \frac{c}{p+1} \right\rfloor$ contribute to the determinant formula since the others result in a recursive call of one of $H(1), \ldots, H(p)$. It is shown in [5] that this yields the following determinant formula:

$$H(n) = \begin{cases} (-1)^{sp} d_p^s d_{-1}^{sp} + K, & \text{if } n = s(p+1) \\ K, & \text{otherwise} \end{cases}$$

with

$$K = \sum_{k=0}^{a} (k+1)(-1)^{n-1-k} d_{n-1-k(p+1)} d_p^k d_{-1}^{n-1-k}.$$

A Maple package for Continuants and Hessenberg Determinants

We implemented the preceding results and will briefly discuss the designed Maple package. After parsing the specification it is tried to identify one of the special cases and return the appropriate formulas, otherwise simply the corresponding recurrence. Display and checking facilities are provided as in the frameforms package.

Specification

Continuants are specified by the triple $d_0, d_1, d_{-1}$of the diagonal generating functions.

The diagonal generating functions of Hessenberg determinants have to be specified in a list of tuples
`[ [pos, function], ... ]`.

Refer to the online help pages or [5] for features concerning the continuants.

Examples

```
> with(HESSENBERGandCONTINUANT):
> Continuant(n,x+y,x,y,print,check);
```

Matrix :

$$\begin{bmatrix} x+y & x & 0 & 0 & o & o \\ y & o & o & 0 & 0 & o \\ 0 & o & o & o & 0 & 0 \\ o & 0 & o & o & o & 0 \\ o & 0 & 0 & o & o & x \\ 0 & o & o & 0 & y & x+y \end{bmatrix}$$

Determinant :

$$\frac{-y^{(n+1)} + x^{(n+1)}}{-y + x}$$

```
> HessenbergDet(n,[ [-1,b],[0,a],[n-3,x],
>    [n-2,y],[n-1,z] ],print,check);
```

Matrix :

$$\begin{bmatrix} a & 0 & 0 & o & o & 0 & x & y & z \\ b & a & 0 & 0 & 0 & 0 & 0 & x & y \\ 0 & b & a & 0 & 0 & 0 & 0 & 0 & x \\ 0 & 0 & b & o & 0 & 0 & 0 & 0 & 0 \\ o & 0 & 0 & o & o & 0 & 0 & 0 & o \\ o & 0 & 0 & 0 & o & o & 0 & 0 & o \\ 0 & 0 & 0 & 0 & 0 & b & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b & a & 0 \\ 0 & 0 & 0 & o & o & 0 & 0 & b & a \end{bmatrix}$$

Determinant :

$$\textit{formula valid for} : , 5 \le n$$

$$a^n + 3\,(-1)^{(n-3)}\, x\, a^2\, b^{(n-3)}$$

$$+2\,(-1)^{(n-2)}\, y\, a\, b^{(n-2)} + (-1)^{(n-1)}\, z\, b^{n-1}$$

Conclusion

We have derived dimensionally parametrized determinant formulas for three determinant classes, the frameforms, the alternants and the continuants and Hessenberg determinants. We described Maple packages that enabled computing the formula of specified determinants of those classes.

The discussed determinant classes and their implementation are excerpts of the author's M.Sc thesis. The Maple packages including online documentation and the thesis offering a more detailed treatise of the topic can be downloaded from the WWW page:

`http://www-tcs.cs.uni-sb.de/mark/det.html`

The book of Metzler [4] is also highly recommended.

References

[1] D. Cox, J. Little, and D. O'Shea. *Ideals, Varieties and Algorithms.* Springer, second edition, 1996.

[2] J. Erickson and R. Seidel. Better lower bounds on detecting affine and spherical degeneracies. *Discrete Computational Geometry*, 13:41–57, 1995.

[3] R. Graham, D. Knuth, and O. Patashnik. *Concrete Mathematics.* Addison–Wesley, 1992.

[4] W. Metzler. *A Treatise on the Theory of Determinants by Thomas Muir.* Dover reprint, 1960.

[5] M. Ziegelmann. Computing dimensionally parametrized determinant formulas. Master's thesis, Universität des Saarlandes, 1997. available at `http://www-tcs.cs.uni-sb.de/mark/det.html`.

Biography

Mark Ziegelmann studied computer science at the universities of Tübingen, Edinburgh and Saarbrücken. He obtained a M.Sc (Diplom) from the Universität des Saarlandes in 1997. Currently, he is a PhD candidate in computer science with a scholarship of the German Science Foundation (DFG). His main research interests are Computational Geometry and Computer Algebra.